Introduction to higher topoi

1: Ascent to ∞-categories

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∞-topoi:

- generalize topoi to "∞-categories"
- are "higher stacks"

Rezk (~1999) Simpson (arXiv 1999)

Toën-Vezzosi, "Higher algebraic geometry: I. topos theory" (2005)

Lurie, "Higher topos theory" (2009) ←

Higher stacks (Grothendieck)

top. space $X \rightsquigarrow$ sheaves of categories on X

Ex: (1) $(U \underset{\text{open}}{\subseteq} X) \longmapsto \mathrm{Bun}_G(U)$ ← groupoid.

(2) $(U \subseteq X) \longmapsto D(U)$ der cat of q Coh sheaves.

↑ scheme

Ex (1): "Categorical sheaf condition": e.g.

$$
\begin{tikzcd}
F(U_1 \cup U_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(U_1) \arrow[d, shift left] \arrow[d, shift right] \\
F(U_2) \arrow[r] & F(U_1 \cap U_2)
\end{tikzcd}
$$

weak pullback of 1-groupoids

eg. Bun_G

generalize to "higher groupoids"

Ex (2): no, wish you could.

Higher groupoids = homotopy theory

G 1-groupoid $\rightsquigarrow$ BG classifying space $\left(\begin{array}{l} \overset{\text{fund gpd}}{\pi_1 BG \simeq G} \\ \pi_k BG = 0 \\ \quad k>1 \end{array} \right)$

Hopf: $B\,\mathrm{Fun}(G,H) \underset{\text{weak equiv.}}{\simeq} \mathrm{Map}(BG, BH)$

"Homotopy hypothesis": $(n \text{ groupoids}) \Leftrightarrow \left(\begin{array}{l} n\text{-truncated} \\ \text{spaces} : \pi_k X = 0 \\ \qquad\qquad\quad k>n \end{array} \right)$

$n \rightsquigarrow \infty \implies$ [simplicial sets, Kan complexes]

Quasicategories: explicit model for

∞-categories (= $(\infty,1)$-categories) (= quasicategory)

(There are other models.)

includes:

∞-groupoids (= $(\infty,0)$-categories)

Boardman-Vogt, "Htpy invt alg str..." (1973)

Joyal, "Quasicategories and Kan complexes" (2002)

Lurie, "Higher topos theory" (2009)

"textbooks"

Cisinski (2019)

Land (2021)

kerodon.net (Lurie)

Riehl-Verity

Simplicial set: functor $X : \Delta^{op} \to Set$

($\Delta \subseteq Cat$ full subcat of $[n] = (0 \to 1 \to 2 \to \cdots \to n)$, $n \geq 0$)

$$sSet := Fun(\Delta^{op}, Set) \ni C = \{C_n\}_{n \geq 0}$$

"n-cells"

ex: Nerve: $Cat \hookrightarrow sSet$

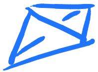
fully faithful

$$C \longmapsto Nerve(C)_n = \left\{ \begin{matrix} \text{functors} \\ [n] \to C \end{matrix} \right\}$$

Quasicategory = simplicial set with "inner horn extensions":

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dotted, "\exists"] & \end{tikzcd}$$

$\forall\ 0 < i < n$



n-simplex

→ $\Delta^n := \mathrm{Hom}_\Delta(-,[n])$ $(0 \le i \le n)$

$\quad \rotatebox{90}{$\subseteq$}$

→ $\Lambda^n_i :=$ largest subobject not containing $d^i : [n-1] \to [n] : x \mapsto \begin{cases} x & x < i \\ x+1 & x \geq i \end{cases}$

— horn $\qquad$ $(n-1)$-cell

$n=2$:

$$\Lambda^2_1 = \left(\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd} \right)$$

$$\Delta^2 = \left(\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] \arrow[ur, phantom, shift right=3ex, "\simeq"] & & 2 \end{tikzcd} \right)$$

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "{(f,g)}"] \arrow[d, hook] & C \\ \Delta^2 \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

→ have composition of morphisms

(not unique)

2-cell - "witness of compositions"

"witness associate"

$n=3$:

In a <u>quasicategory C</u>:

0-cells - "objects"

1-cells - "morphisms"

2-cells - "witness composition"

$\vdots$ $\qquad$ $\vdots$

$$x \in C_0 \mapsto 1_x \in C_1$$

(quasicat C is a 1-category iff inner horn extensions are unique)

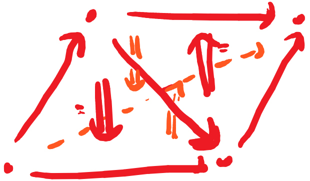
<u>functor</u> of q.cats: $C \to D$ map of sSets.

Example: quasicategory Cat_1 of 1-categories

0-cell: C cat

1-cell: $C_0 \xrightarrow{F} C_1$ functors

2-cell:
$$\begin{tikzcd} & C_1 \arrow[dr, "G"] & \\ C_0 \arrow[ur, "F"] \arrow[rr, "H"'] & \arrow[u, phantom, "\Downarrow"] & C_2 \end{tikzcd}$$
not iso

3-cell: 

"commutes"

"Duskin nerve"

$$\left\{ C \xrightarrow{\text{functors}} \mathrm{Cat}_1 \right\} \Longrightarrow \left\{ C \xrightarrow{\text{pseudofunctor}} \mathrm{Cat} \right\}$$

1-category 1-category

Example:

Homotopy theory $\rightsquigarrow$ ∞-category

$C \supset W$ $\rightsquigarrow$ $L(C,W)$ - ∞-cat

1-cat

collection of "weak equivalences"

(e.g. a Quillen model category)

$$L(C,W) \longrightarrow C[W^{-1}]$$

$C[W^{-1}]$ 1-cat

"fraction"

Basic notions:

C quasicat

Isomorphism in C: $f: x \to y \in C_1$ s.t. $\exists\, g: y \to x \in C_1$ and $g \circ f \underset{\exists}{\sim} 1_x$, $f \circ g \underset{\exists}{\sim} 1_y$

∞-groupoid: all morphisms are iso

$\iff$ Kan complexes

$\left(\exists \text{ horn ext } \forall \begin{array}{ccc} \Lambda^n_i & \longrightarrow & C \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array} \quad \underline{0} \le i \le \underline{n} \right)$

homotopy thy

Joyal (2002)

functor quasicategory: $\mathrm{Fun}(C,D)$ - internal function cpx in $sSet = \mathrm{Fun}(\Delta^{op}, Set)$

$$\mathrm{Fun}(C,D)_n = \{ C \times \Delta^n \dashrightarrow D \}$$

0-cells = functors

1-cells = nat. transf.

is a quasicategory if C, D are (non-trivial)

$\Rightarrow$

- natural isos of functors
- equiv of quasicategories

$$\cdot \; C \rightleftarrows D, \quad \simeq \quad \simeq$$

Join: 1-cats C, D $\rightsquigarrow$ 1-cat $C \subseteq \underline{C * D} \supseteq D$

$(C*D)_0 = C_0 \amalg D_0$, $\quad (C*D)_1 = C_1 \amalg \underbrace{C_0 \times D_0}_{\{c \to d\}} \amalg D_1$

left/right cones: $\Delta^0 * C$, $C * \Delta^0$ $\Leftarrow$ left/right cones

extend to s.Set: $\underline{\Delta}^p * \underline{\Delta}^q = \Delta^{p+1+q}$ $\quad (p, q \geq -1, \ \Delta^{-1} = \emptyset)$

Join of q.cats is a q.cat!

Slice: $\mathcal{C},\ x \in \mathcal{C}_0 \rightsquigarrow \mathcal{C}_{/x}$

$$\left\{ T \dashrightarrow \mathcal{C}_{/x} \right\} \Longleftrightarrow \left\{ \begin{tikzcd} \Delta^0 \arrow[r, "x"] \arrow[d] & \mathcal{C} \\ T \star \Delta^0 \arrow[ur, dashed] & \end{tikzcd} \right\}$$

so $(\mathcal{C}_{/x})_0 = \{ c \to x \in \mathcal{C}_1 \}$ $\qquad \mathcal{C}$ q.cat

$$(\mathcal{C}_{/x})_1 = \left\{ \begin{tikzcd} & c_1 \arrow[dr] & \\ c_0 \arrow[ur] \arrow[rr] & & x \end{tikzcd} \in \mathcal{C}_2 \right\} \Longrightarrow \mathcal{C}_{/x} \text{ q.cat}$$

$\vdots \quad \vdots$

Terminal object: $x \in C_0$ s.t.

C q.cat

$\Updownarrow$

$\Delta^1 = (\cdot \to \cdot)$

$C_{/x} \xrightarrow{\text{forget}} C$ is an equivalence of quasicats.

equivalently:

$$\begin{tikzcd} \mathrm{Map}_C(y,x) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "(ev_0, ev_1)"] \\ \Delta^0 \arrow[r, "(y,x)"'] & C \times C \end{tikzcd}$$

∞-groupoid.

"mapping space"

x terminal $\iff$ $\mathrm{Map}_C(y,x) \to \Delta^0$ equiv $\forall\, y \in C_0$

(general)

Slice: $f : J \to C \rightsquigarrow C_{/f} \in$

$$\{ T \dashrightarrow C_{/f} \} \iff \left\{ \begin{tikzcd} J \arrow[r, "f"] \arrow[d] & C \\ T * J \arrow[ur, dashed] & \end{tikzcd} \right\}$$

$J = \Delta^0$

$\Rightarrow C_{/x}$

Limit cone:

$$\begin{tikzcd} J \arrow[r, "f"] \arrow[d] & C \\ \underline{\Delta^0 * J} \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

s.t. $C_{/\hat{f}} \xrightarrow{\text{forget}} C_{/f}$ is equiv. of quasicats

$(\Longleftarrow)$ $\hat{f}$ corr $\Delta^0 \to C_{/f} \in (C_{/f})_0$ lim $\in$ terminal in $(C_{/f})_0$

Example: sheaves!

X - topological space,

presheaf:

$$\begin{tikzcd} & S \subseteq I \arrow[r, mapsto] & \bigcap_{i\in S} U_i & & \\ P_{fin}(I) \arrow[rrr, "\alpha_{\mathcal{U}}"'] & & & \mathrm{Open}_X^{op} \arrow[r, "F"] & \mathcal{C} \end{tikzcd}$$

1-cat $\quad$ ∞-cat

$P_{fin}(I)$ is $\Delta^{\circ} * P^{\circ}_{fin}(I)$

$\mathcal{U} = \{U_i\}_{i\in I}$, $U_{\emptyset} = \bigcup U_i$

F is a **sheaf** if: $F\circ\alpha_{\mathcal{U}}$ is a limit cone $\forall \{\mathcal{U}_i\}$

(Exc: $\mathcal{C} = \mathrm{Cat}_1$)

∞-category of ∞-categories

$$\begin{array}{ccc} \mathrm{Cat} \hookrightarrow q\mathrm{Cat} & \longrightarrow & \mathrm{Cat}_\infty \\ \downarrow & & \\ \mathrm{sSet} & & \end{array}$$

1-categories

$\uparrow$ quasicategory ("large")

Lurie.

Cisinski-Nguyen

Riehl-Verity

"straightening/unstraightening"

Correspondence:

$$\left\{ \begin{array}{c} E \\ \downarrow \\ C \end{array} \text{ Grothendieck opfibration} \right\} \Longleftrightarrow \left\{ C \dashrightarrow Cat \right\}$$

1-categories

pseudofunctors

1-cat

$\cong$

functor

$$\left\{ C \longrightarrow Cat_1 \right\}$$

2-cat

Correspondence:

quasicats

$f(x)$

$$\left\{ \begin{array}{c} E \\ \downarrow \\ C \end{array} \text{ coCartesian fibration} \right\} \Longleftrightarrow \left\{ C \overset{f}{\dashrightarrow} Cat_\infty \right\}$$

$|x|$

?

functors

(up to $E \xrightarrow{\text{equiv}} E'$ over C)

(up to nat iso)

"characterizes" Cat_∞ up to equivalence ← "large"

(in fact $Cat_\infty \simeq \bigcup_\kappa Cat_\infty^\kappa$)

large

ess. small

∞-category of ∞-groupoids : $\quad \mathcal{S} \subseteq \mathrm{Cat}_\infty$

$$\left\{ \begin{array}{c} E \\ \downarrow \\ C \end{array} \;\; \begin{array}{l} \text{coCartesian} \\ \text{fibrations,} \\ \text{fibers are} \\ \infty\text{-groupoids} \end{array} \right\} \Longleftrightarrow \left\{ \underset{\text{functors}}{C \dashrightarrow \mathcal{S}} \right\}$$

Fact: any $D \to C$ btwn ∞-grpds is $\simeq$ to a coCart fib:

$$\mathcal{S}_{/C} \simeq \mathrm{Fun}(C^{op}, \mathcal{S})$$

$c \in \mathcal{S}_0$

Warning:

$$\underset{\text{1-cat}}{\overset{}{q\mathrm{Cat}}} \longrightarrow \mathrm{Cat}_\infty$$

1-cat $\quad$ quasicat

∞-category theory often involves interpolating between:

1) constructions in $q\mathrm{Cat}$
2) constructions in Cat_∞

Example:

$$q\mathrm{Cat}^{op} \times q\mathrm{Cat} \xrightarrow{\mathrm{Fun}} q\mathrm{Cat} \quad \overset{??}{\Longrightarrow} \quad \mathrm{Cat}_\infty^{op} \times \mathrm{Cat}_\infty \xrightarrow{\mathrm{Fun}\ ??\ \checkmark} \mathrm{Cat}_\infty$$

Barwick-Shah, "Fibrations in ∞-category theory" (arXiv 2016)

Yoneda, etc.

C quasicategory

$$h_C : C^{op} \times C \longrightarrow \mathcal{S} \quad \text{classifies} \qquad \begin{array}{c} Tw(C) \\ \big\downarrow \\ C^{op} \times C \end{array} \begin{array}{l} \text{twisted arrow cat} \\ \\ \text{– coCart fib} \\ \\ \end{array}$$

so $h_C(x,y) \simeq \mathrm{Map}_C(x,y)$
$x, y \in C_0$

$$Tw(C)_n := C_{n+1+n} = \{ (\Delta^n)^{op} * \Delta^n \longrightarrow C \}$$

$$Tw(C)_0 = \{ \cdot \xrightarrow{f} \cdot \in C_0 \} \qquad Tw(C)_1 = \left\{ \begin{tikzcd} \cdot \arrow[r, "f"] & \cdot \arrow[d] \\ \cdot \arrow[u] \arrow[ur] \arrow[r, "g"'] & \cdot \end{tikzcd} \in C_3 \right\}$$

etc.

Yoneda embedding of ∞-categories:

$$y: C \hookrightarrow \mathrm{Fun}(C^{op}, S) \quad \Longleftrightarrow \quad h: C^{op} \times C \longrightarrow S$$

Adjoint functors of ∞-cats:

via

$$C \underset{g}{\overset{f}{\rightleftarrows}} D \,, \qquad h(f(-),-) \simeq h(-,g(-))$$

nat iso of $C^{op} \times D \to S$

Lurie, Cisinski

∞-categories $= (\infty,1)$-categories : are not really "higher category theory"

"category + homotopy"

(∞,n)-categories ? Yes

Going from 1-categories $\rightsquigarrow$ ∞-categories:

- "The square $\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, "\text{"}\ \text{"}"] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$ commutes" in $\mathcal{C}$

I have not told you what this is.

Famous example:

in $\mathcal{S}$

"loops"

$$\begin{tikzcd} \Omega X \arrow[r, "p"] \arrow[d, "p"'] & 1 \arrow[d, "f"] \\ 1 \arrow[r, "f"'] & X \end{tikzcd}$$

- truly comm square

$\not\Downarrow$

- pull back

- notions of **finite** ∞-category, **finite** limit/colimit

  ⇒ C is finite complete iff $\begin{cases} (1) \text{ has term obj} \\ (2) \text{ has pullbacks} \end{cases}$

  ⇒ $f : C \to D$ preserves finite limits iff preserves these
  (C, D finite complete)

- **Warning** finite 1-cat $\not\subset$ finite ∞-cat

  finite group $G \neq$ Sel , not finite as ∞-cat.
  (BG is ∞-dim'l)

- $f : X \to Y$ monomorphism if
$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
pullback (thr comm)

$\Longleftrightarrow$ $\Delta(f) : X \to X \times_Y X$ iso (if finite complete)

Example: In $\mathcal{S}$ = ∞-groupoids = "htpy theory of spaces"

$f : X \rightarrowtail Y$ mono iff $X \xrightarrow{\text{equiv}} X' \subseteq Y$

union of some path components

("fully faith. functor of ∞-groupoids").

Warning: gf mono

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, tail, bend right, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

$\not\Rightarrow$ f mono

$\Rightarrow$ in particular: $\Delta(f)$ might not be mono.

However: gf mono, g mono $\Rightarrow$ f mono

- $f : X \to Y$ is **n-truncated** if $\Delta^{n+2}(f)$ is iso

$$\mathrm{Iso} \subseteq \underset{\sim -1}{\mathrm{Mono}} \subseteq \mathrm{Trunc}_0 \subseteq \mathrm{Trunc}_1 \subseteq \mathrm{Trunc}_2 \subseteq \cdots \subseteq C_1$$

Example: $X \xrightarrow{f} *$ in $\underset{\omega\text{-gpd}}{\mathcal{S}}$ is n-truncated iff

$\pi_k X = 0 \quad k > n$

$\mathcal{S}_{\leq n} \subseteq \mathcal{S}$ full subcat of n-truncated objects "n-groupoids"

* Dual notions: epimorphisms, n-cotruncated morphism

Weird (in standard examples like $\underline{S}$)

Fact: $X \to * \in \underline{S}$ is epimorphism iff

X acyclic: $H_* X \cong H_*(\text{point})$.

$\Longrightarrow$ no epi/mono factorization in $\underline{S}$

Raptis, "Some characterizations of acyclic maps" (2019)

- Equalizers/coequalizers are a kind of finite limit/colimit

$$(\bullet \rightrightarrows \bullet) \quad \rightsquigarrow \quad (\bullet \rightrightarrows \bullet \Rrightarrow \bullet \equiv \bullet \equiv \cdots)$$

$\wr|$ generate Δ, (or Δ^{op})

<u>Next time:</u> ∞-topoi